A norm-based approach to the minimum-energy multivariable perfect control design

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***Abstract*—In this paper a minimum-energy problem of perfect control algorithm dedicated to LTI MIMO plants described in the discrete-time state-space framework is shown. Following the new facts concerning the above-mentioned issue, a new solution based on the mathematical norms utilizing some generalized inverse formulas is given in this paper. The simulation instance confirms the correctness of a new intriguing method, which certainly leads to the final proof of whole, undiscovered yet, modern control theory.**

***Index Terms*—perfect control, minimum-energy control design, generalized σ-inverse, LTI MIMO, state-space domain, norms, Neumann's theorem**

I. Introduction

The perfect control strategy is well-known and often found in the modern control engineering practice [1]–[6]. Notwithstanding, the control algorithm seems to only be clear for square multi-input/multi-output (MIMO) plants including the single-input/single output (SISO) ones [7]. It is true, for such set of systems the problem of inverse model control (IMC) does not rise doubts due to existence of so-called uniqueness during design of perfect control schemes. However, in case of nonsquare systems having different number of input and output variables the behavior of nonuniqueness introduces the expected complexity [7], [8]. In fact, an application of generalized inverses with arbitrary degrees of freedom may cause difficulties even for experienced designer. Due to selection of proper inverse we can improve the properties of the perfect control strategy dedicated, in particular, to linear time-invariant (LTI) MIMO discrete-time systems described in the state-space framework [7]. The aforementioned features should be understood in terms of, e.g., minimum-energy of the control inputs or, in more general, in context of improvement of robustness behavior [8]. Following the recently obtained results we can state that the classical minimum-norm inverse based on the Moore-Penrose assumptions does not guarantee the minimal energy for right-invertible plants [8]–[10]. The heuristic and analytical results have shown that in case of systems with greater number of input than output variables we should apply other inverses, for instance the quite recently introduced σ and H ones [7]. These inverses can not only stabilize entire perfect control systems but, importantly, can provide their minimum-energy characteristic. It should be strongly emphasized that this paradigm can not be carried out for left-invertible plants, in such a case the said control strategy does not exist. In this paper the newly introduced norm-based approach to the minimum-energy perfect control problem is presented, which undoubtedly sheds a new light on the complex problem examined here.

Following the introduction section in another one the system representation of analyzed in the paper multivariable plants is shown. In the same unit the perfect control algorithm is presented. The energy-based approach to the perfect control law is given in the next section. A main goal of this paper related to the norm-based solution is also presented in the third section. Section four shows representative simulation example, which confirms the usefulness of postulated within the paper a new method. At the end the final conclusions are tabelarized.

II. Perfect control algorithm

We start our considerations with the definition of plants which are touched in the paper.

A. System representation

We consider the LTI MIMO controllable systems described in the discrete-time state-space framework as follows

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k), \end{aligned} \tag{1}$$

having initial condition $\mathbf{x_0} = \mathbf{x}(0)$ and $\mathbf{x}(k) \in \mathbb{R}^n$, $\mathbf{y}(k) \in \mathbb{R}^{n_y}$, $\mathbf{u}(k) \in \mathbb{R}^{n_u}$. Additionally we assume that the parameter matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are of appropriate dimensions whilst k denotes a discrete time.

B. Perfect control law

The perfect control for right-invertible ($n_u \geq n_y$) plants defined by Eqs. (1) sounds as follows [8]

$$\mathbf{u}(k) = -(\mathbf{CB})^{\mathrm{R}}\mathbf{CAx}(k), \tag{2}$$

where symbol $(.)^{\mathrm{R}}$ describes any generalized inverse of the parameter matrix $\mathbf{CB}$. Note that above-mentioned control strategy can be immediately obtained in terms of application of predictor theory for reference value/setpoint $\mathbf{y_{ref}}(k) = \mathbf{0}$. It guarantees that after time delay d the output is equal to the above-mentioned setpoint. Additionally, the Eq. (2) can be applied in the minimum variance control scenario.

Remark 1: It should strongly be emphasized that our perfect control algorithm is different than the well-known optimal control issue. For details see Ref. [8].

Having already the fundamental issues let us start our main considerations below.

III. Energy-based paradigm of perfect control input variables

Observe that after application of formula (2) directly to the Eqs. (1) we can propose the crucial relation in the following manner

$$\mathbf{x}(k+1) = \left[(\mathbf{I_n} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}\mathbf{C})\mathbf{A}\right]\mathbf{x}(k), \tag{3}$$

where $\mathbf{I_n}$ represents the identity n-matrix. Moreover, after some manipulations we receive a general solution in the form of

$$\mathbf{x}(k) = \left[(\mathbf{I_n} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}\mathbf{C})\mathbf{A}\right]^k\mathbf{x}(0). \tag{4}$$

Remark 2: Noticing that through the Eq. (4) we can calculate a value of $\mathbf{x}(k)$ for $k = 1, \ldots, N-1$, where N denotes a time horizon, based on the initial state vector $\mathbf{x}(0)$ only. This intriguing observation will be helpful during further energy-based research shown in this paper.

A. Energy-based approach to the perfect control design

It is important that energy of the perfect control inputs can be presented in every instant of time k as follows

$$E(k) = \langle \mathbf{u}(k), \mathbf{u}(k)\rangle = \|\mathbf{u}(k)\|^2, \tag{5}$$

where symbols $\langle \mathbf{a}, \mathbf{b}\rangle$ and $\|\mathbf{x}\|$ denote the scalar product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and a norm of vector $\mathbf{x}(k)$, respectively.

Now, after substitution the Eq. (4) strictly to the formula (2) we obtain (for brevity we assume that $\mathbf{y_{ref}}(k) = \mathbf{0}$)

$$\mathbf{u}(k) = -(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\left[(\mathbf{I_n} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}\mathbf{C})\mathbf{A}\right]^k\mathbf{x}(0), \tag{6}$$

and finally the energy of the perfect control inputs $E(k)$ sounds as follows

$$E(k) = \|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\left[(\mathbf{I_n} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}\mathbf{C})\mathbf{A}\right]^k\mathbf{x}(0)\|^2. \tag{7}$$

Observe, that for simplicity we can put without loss a generality

$$\mathbf{P} = \mathbf{I_n} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}\mathbf{C}, \tag{8}$$

and based on Eq. (7) we can immediately receive

$$E(k) = \|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\,(\mathbf{PA})^k\,\mathbf{x}(0)\|^2. \tag{9}$$

Hence the entire energy is equal to

$$E = \sum_{k=0}^{\infty} E(k). \tag{10}$$

So, the main issue arises: what kind of generalized inverse of the $\mathbf{CB}$ product should be chosen in order to guarantee the minimum-energy perfect control inputs of plants defined by Eqs. (1)? This fundamental problem is explained in detail in the next section. For special selected so-called degrees of freedom involved in parameter matrix β of the σ-inverse of form [7], [8]

$$\mathbf{G}^{\mathrm{R}}_{\sigma} = \beta^{\mathrm{T}}([\mathbf{G}\beta^{\mathrm{T}}])^{-1}, \tag{11}$$

we receive smaller energy than in case of application the classical minimum-norm one, in general.

B. Estimation of minimal energy

In order to determine the entire energy E let us describe its consumption E_t after t-steps as follows

$$E_t = \sum_{k=0}^{t} E(k), \quad t = 1, 2, 3\ldots \tag{12}$$

For such assumption, after using the well-known norm peculiarities, we can formulate the following approximation (for example similar estimations are presented in Refs. [11], [12])

$$\begin{aligned} E_t &= \sum_{k=0}^{t}\|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\,(\mathbf{PA})^k\,\mathbf{x}(0)\|^2 \\ &\geq \frac{1}{t+1}\left(\sum_{k=0}^{t}\|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\,(\mathbf{PA})^k\,\mathbf{x}(0)\|\right)^2. \end{aligned} \tag{13}$$

Thus,

$$E_t \geq \frac{1}{t+1}\left(\left\|\sum_{k=0}^{t}(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\,(\mathbf{PA})^k\,\mathbf{x}(0)\right\|\right)^2, \tag{14}$$

and finally

$$E_t \geq \frac{1}{t+1}\left\|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\left[\sum_{k=0}^{t}(\mathbf{PA})^k\right]\mathbf{x}(0)\right\|^2. \tag{15}$$

Of course, the aforementioned results based on the form $\|\mathbf{F}\|$ being the inductive/operator norm of any matrix $\mathbf{F}$, i.e., $\|\mathbf{F}\| = \max_{\|\mathbf{x}\|\leq 1}\{\|\mathbf{Fx}\| : \mathbf{x} \in \mathbb{R}^n\}$. In order to further estimation of the minimal perfect control energy we should apply the lemma given below.

Lemma 1 (Neumann): Let the matrix $\mathbf{F}$ fulfills the condition $\|\mathbf{F}\| < 1$. Then, a following statement holds

$$\|(\mathbf{I} - \mathbf{F})^{-1}\| \leq \frac{1}{1 - \|\mathbf{F}\|}. \tag{16}$$

Next, assuming that $\|\mathbf{PA}\| < 1$ we obtain

$$\begin{aligned} \sum_{k=0}^{t}(\mathbf{PA})^k &= \mathbf{I_n} + \mathbf{PA} + (\mathbf{PA})^2 + \ldots + (\mathbf{PA})^t \\ &= (\mathbf{I_n} - \mathbf{PA})^{-1}\left[\mathbf{I_n} - (\mathbf{PA})^{t+1}\right], \end{aligned} \tag{17}$$

and finally

$$E_t \geq \frac{1}{t+1}\left\|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}(\mathbf{I_n} - \mathbf{PA})^{-1}\left[\mathbf{I_n} - (\mathbf{PA})^{t+1}\right]\mathbf{x}(0)\right\|^2. \tag{18}$$

Note, if the matrix $\mathbf{F}$ is nonsingular we obtain the following approximation

$$\|\mathbf{x}\| = \|\mathbf{F}\mathbf{F}^{-1}\mathbf{x}\| \leq \|\mathbf{F}^{-1}\|\|\mathbf{F}\mathbf{x}\|, \tag{19}$$

and therefore

$$\|\mathbf{F}\mathbf{x}\| \geq \frac{1}{\|\mathbf{F}^{-1}\|}\|\mathbf{x}\|. \tag{20}$$

Moreover, because of $\|(\mathbf{PA})^{t+1}\| < 1$ we can state that the matrix inverse of $[\mathbf{I_n} - (\mathbf{PA})^{t+1}]$ certainly exists. Thus

$$\begin{aligned} E_t &\geq \frac{1}{t+1}\left\|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}(\mathbf{I_n}-\mathbf{PA})^{-1}\left[\mathbf{I_n}-(\mathbf{PA})^{t+1}\right]\mathbf{x}(0)\right\|^2 \\ &\geq \frac{1}{t+1}\frac{\|\mathbf{x}(0)\|^2}{\left\|[\mathbf{I_n}-(\mathbf{PA})^{t+1}]^{-1}(\mathbf{I_n}-\mathbf{PA})\left[(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\right]^{\mathrm{R}}\right\|^2} \\ &\geq \frac{1}{t+1}\frac{\left\|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\right\|^2\|\mathbf{x}(0)\|^2}{\|[\mathbf{I_n}-(\mathbf{PA})^{t+1}]^{-1}\|^2\|\mathbf{I_n}-\mathbf{PA}\|^2}. \end{aligned} \tag{21}$$

After using the Neumann's lemma we obtain the interesting estimation

$$E_t \geq \frac{\left(1-\|\mathbf{PA}\|^{t+1}\right)^2}{t+1}\frac{\left\|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\right\|^2}{\|\mathbf{I_n}-\mathbf{PA}\|^2}\|\mathbf{x}(0)\|^2. \tag{22}$$

It confirms now, that for $\mathbf{x}(0) = \mathbf{0}$ the energy $E_t = 0$.

Note, that $\frac{\left(1-\|\mathbf{PA}\|^{t+1}\right)^2}{t+1} \to 0$ as $t \to \infty$, so the total energy of the perfect control inputs does not increase in an unlimited way and it can be close to zero even when the vector $\mathbf{x}(0) \neq \mathbf{0}$. Based on the function

$$e(t) = \frac{\left(1-\|\mathbf{PA}\|^{t+1}\right)^2}{t+1}, \tag{23}$$

the energy E_t in the first seconds increases, then it becomes smaller. The more $\|\mathbf{PA}\|$ is close to 1, the more time it has to pass before the energy begins to decrease. The $\|\mathbf{PA}\|$ is close to zero, the faster this moment is and the effect is unnoticeable for certain values of $\|\mathbf{PA}\|$.

Based on the conducted considerations we can state that the plants described by Eqs. (1) is stable if the following formula holds

$$\|\mathbf{PA}\| < 1. \tag{24}$$

In this scenario the crucial energy is the smallest. On the other hand, if the condition (24) does not occur, the system can be stable in spite of this (it is enough to assume that all eigenvalues of the matrix $\mathbf{PA}$ have modulus strictly less than one). However, in this case, determining a similar estimation of the minimum energy is very difficult.

C. The conditions related to the minimization of energy

Let us now try to estimate the said energy E_t from the upper limit. This operation gives us the opportunity to obtain the minimal energy of the perfect control input variables in respect to the special chosen degrees of freedom of our stable σ-inverse defined by Eq. (11). Then, we have

$$\begin{aligned} E_t &= \sum_{k=0}^{t}\|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\,(\mathbf{PA})^k\,\mathbf{x}(0)\|^2 \\ &\leq \sum_{k=0}^{t}\|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\|^2\|\mathbf{PA}\|^{2k}\|\mathbf{x}(0)\|^2 \\ &= \|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\|^2\sum_{k=0}^{t}\|\mathbf{PA}\|^{2k}\|\mathbf{x}(0)\|^2. \end{aligned} \tag{25}$$

Because of

$$\sum_{k=0}^{t}\|\mathbf{PA}\|^{2k} = \frac{1-\|\mathbf{PA}\|^{2k+1}}{1-\|\mathbf{PA}\|^2}, \tag{26}$$

we have

$$E_t \leq \frac{\|(\mathbf{CB})^{\mathrm{R}}\mathbf{CA}\|^2\left(1-\|\mathbf{PA}\|^{2t+1}\right)}{1-\|\mathbf{PA}\|^2}\|\mathbf{x}(0)\|^2. \tag{27}$$

Next, assuming that $\|\mathbf{PA}\| \geq 0$, we can write the following formula

$$E \leq \frac{\|(\mathbf{CB})^{\mathrm{R}}\|^2\|\mathbf{C}\|^2\|\mathbf{A}\|^2}{1-\|\mathbf{PA}\|^2}\|\mathbf{x}(0)\|^2, \tag{28}$$

which means that the total perfect control energy as in Eq. (2) of plants defined by Eqs. (1) is bounded. Therefore, we can state that we can select such inverse $(\mathbf{CB})^{\mathrm{R}}$, for which our energy will be the smallest. For that reason, let us try to propose such conditions which certainly are in relation with the following equation

$$\begin{aligned} &\frac{\|(\mathbf{CB})^{\mathrm{R}}\|^2\|\mathbf{C}\|^2\|\mathbf{A}\|^2}{1-\|\mathbf{PA}\|^2} = \frac{\|(\mathbf{CB})^{\mathrm{R}}\|^2\|\mathbf{C}\|^2\|\mathbf{A}\|^2}{1-\|(\mathbf{I_n}-\mathbf{B}(\mathbf{CB})^{\mathrm{R}}\mathbf{C})\mathbf{A}\|^2} \leq \\ &\frac{\|(\mathbf{CB})^{\mathrm{R}}\|^2\|\mathbf{C}\|^2\|\mathbf{A}\|^2}{1-\left(1+\|\mathbf{B}\|^2\|(\mathbf{CB})^{\mathrm{R}}\|^2\|\mathbf{C}\|^2\right)\|\mathbf{A}\|^2}. \end{aligned} \tag{29}$$

In order to obtain the pursued solution let us calculate the minimum of the subsequent function

$$f(\tau) = \frac{a^2c^2\tau^2}{1-(1+b^2c^2\tau^2)a^2}, \tag{30}$$

where $\tau = \|(\mathbf{CB})^{\mathrm{R}}\|$, $a = \|\mathbf{A}\|$, $b = \|\mathbf{B}\|$, $c = \|\mathbf{C}\|$.

Observe, that based on the crucial formula

$$f'(\tau) = \frac{2a^2c^2\left(1-a^2\right)\tau}{\left(a^2\left(b^2c^2\tau^2+1\right)-1\right)^2}, \tag{31}$$

we can state that the total energy E can be obtained in case of $\|\mathbf{A}\| < 1$. Additionally, the minimal energy is related to the value of norm $\|(\mathbf{CB})^{\mathrm{R}}\|$.

For $\|\mathbf{A}\| = 1$ it does not matter which inverse will be used. In such a case the energy will be the same for any arbitrary chosen generalized right inverse of $\mathbf{CB}$ in form of $(\mathbf{CB})^{\mathrm{R}}$.

Only when $\|\mathbf{A}\| > 1$, the energy depends on a type of inverse is used.

In order to illustrate the correctness of the new method let us examine an intriguing simulation instance in the next section.

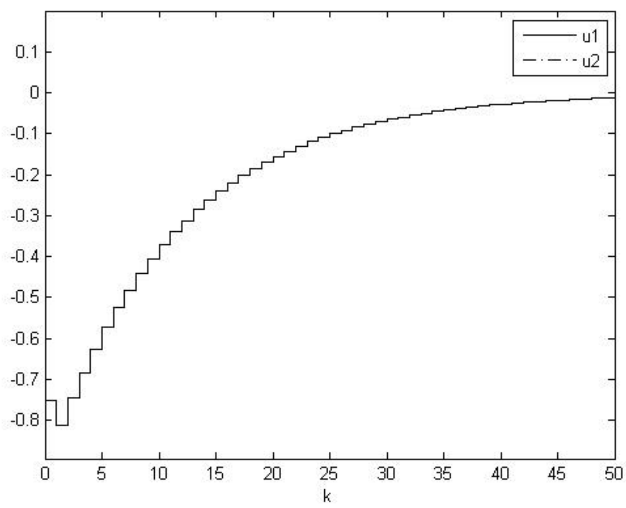

Fig. 1. Perfect control runs of $\mathbf{u}(k)$, case: minimum-norm inverse.

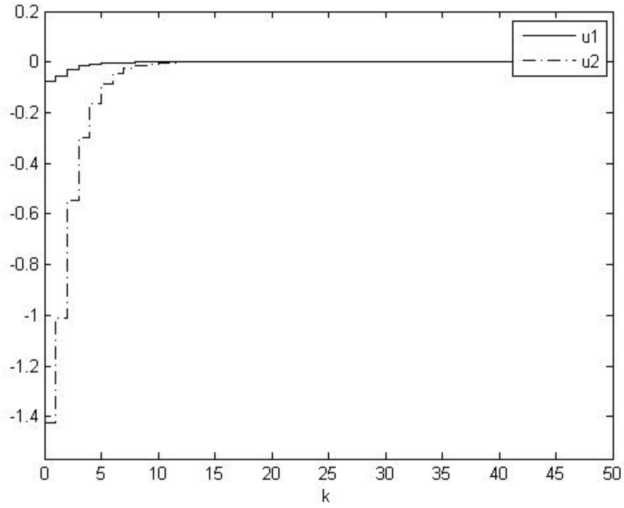

Fig. 2. Perfect control runs of $\mathbf{u}(k)$, case: σ-inverse.

As for plants with $\|\mathbf{A}\| < 1$ the formulas (30) and (31) are not quite representative, we will rather apply a system having $\|\mathbf{A}\| > 1$. In the former case it does not matter what kind of inverse is used. For both unique minimum-norm inverse application and non-unique σ-inverse employment we obtain the smallest value of energy E since $\|(\mathbf{CB})^{\mathrm{R}}\| \geq \frac{1}{\|\mathbf{CB}\|}$, in general.

IV. Simulation example

Consider the system described by Eqs. (1) with $\mathbf{A} = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0.33 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 0 & 10 \end{bmatrix}$ and $\mathbf{x_0} = \begin{bmatrix} 3 & 1 \end{bmatrix}$. Then, for $\beta = \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}$, we receive the formula as in Eq. (28) with appropriate components (not presented here due to a complexity). So, after standard mathematical manipulations in Mathematica environment our solution sounds as follows

$$\beta = \begin{bmatrix} 0.054 & 1 \end{bmatrix}. \tag{32}$$

Now, the total energy obtained through an application of σ-inverse with selected β to the product of $\mathbf{CB}$ (see Eq. (2)) is equal to $E^{\sigma}_{100} = 3.480$. On the other hand, usage of the classical minimum-norm right inverse brings us the higher control energy equals $E_{100} = 9.391$. The perfect control runs are depicted in figures below. Note that after time delay $k \geq d = 1$ (see Eqs. (1) and (2)) the output remains at the reference/setpoint $y_{ref}(k) = 0$.

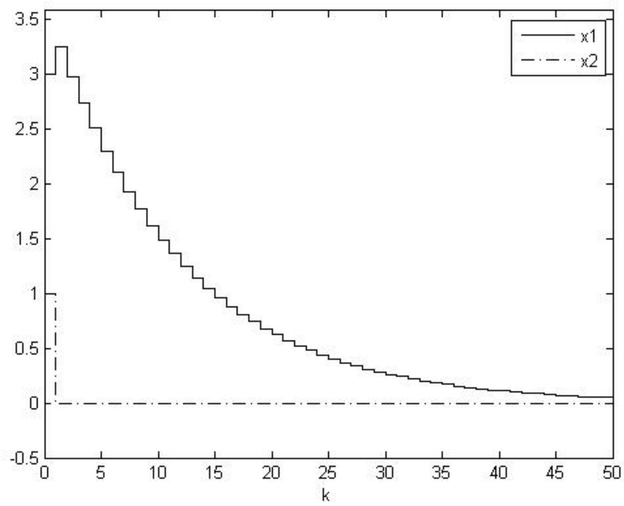

Fig. 3. Perfect control runs of $\mathbf{x}(k)$, case: minimum-norm inverse.

V. Conclusions

In this paper an attempt to minimization of the perfect control input energy is presented. The solution based on the approximation technique involves a norm mechanism. The new approach strictly related to the analytical paradigm has successfully been confirmed by the fruitful simulation example made in Mathematica environment. The further study will be focused on the application of the method in the practical problems.

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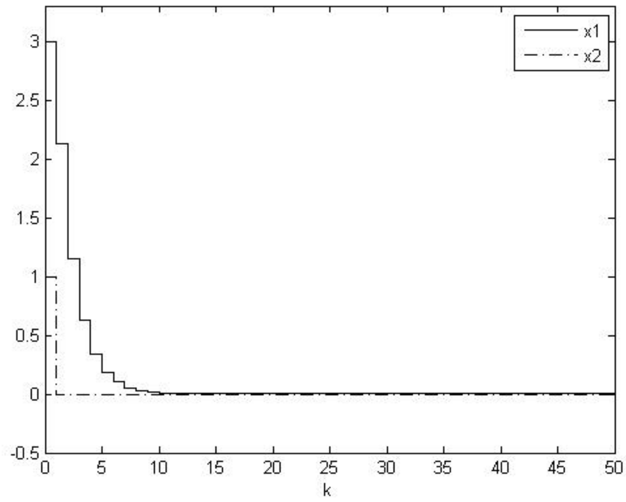

Fig. 4. Perfect control runs of $\mathbf{x}(k)$, case: σ-inverse.

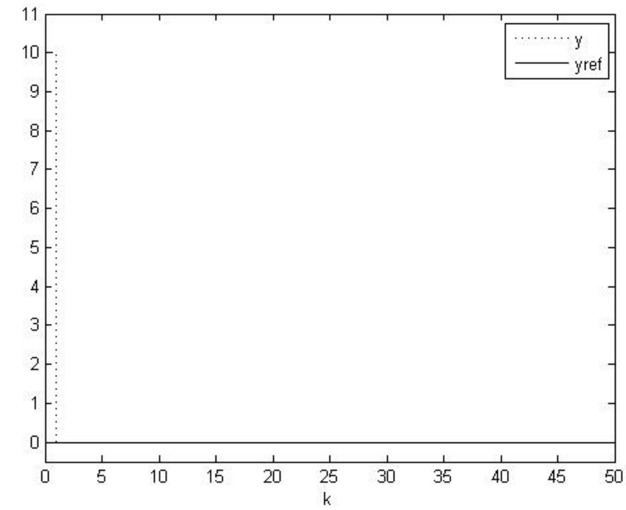

Fig. 5. Perfect control runs: $y(k)$ vs. $y_{ref}(k) = 0$, both cases.

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